1D Coulomb problem with deformed Heisenberg algebra

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Symmetry in Nonlinear Mathematical Physics
June, 20-26, Kyiv

In this report we consider one dimensional quantum mechanics with the following deformation

$$[X, P] = i(1 + \beta P^2). \tag{1}$$

Such a deformation implies that there exists minimal resolution of length $\Delta X \geq \sqrt{\beta}$, i.e., there is no possibility to measure coordinate X with accuracy more then ΔX. It is expected that deformation can help to avoid divergences.

We solve eigenvalue equation for Coulomb-type potential

$$P^2\psi - \frac{\alpha}{X}\psi = E\psi. \tag{2}$$

Momentum representation

$$P = p, \qquad X = i(1+\beta p^2)\frac{d}{dp}. \tag{3}$$

We have to redefine scalar product to ensure Hermicity of X:

$$\langle\phi|\psi\rangle = \int\limits_{-\infty}^{\infty} \frac{\phi^*(p)\psi(p)}{1+\beta p^2}dp, \qquad \langle\phi|A\psi\rangle = \int\limits_{-\infty}^{\infty} \frac{\phi^*(p)}{1+\beta p^2}A\psi(p)dp. \tag{4}$$

It is naturally to demand

$$X\frac{1}{X} = 1 \tag{5}$$

Multiplication of eigenvalue equation gives

$$XP^2\psi - \alpha\psi = EX\psi. \tag{6}$$

Solution of this equation reads

$$\psi_\epsilon(p) = \frac{C_\epsilon}{\epsilon + p^2} e^{\left(\frac{i\alpha}{1-\epsilon\beta}\left(\frac{1}{\sqrt{\epsilon}}\,\mathrm{arctan}\,\frac{p}{\sqrt{\epsilon}} - \sqrt{\beta}\,\mathrm{arctan}\,\sqrt{\beta}p\right)\right)}, \tag{7}$$

where $\epsilon = -E$.

Properties of eigenfunctions:

- if $\epsilon > 0$ then $\langle\psi_\epsilon|\psi_\epsilon\rangle$ exists.

Spectrum

Integrating of equation (6) gives

$$p^2\psi(p) + i\alpha \int\limits_{-\infty}^{p} \frac{\psi(q)dq}{1+\beta q^2} + c[\psi] = E\psi(p). \tag{8}$$

It looks like as an eigenvalue equation. Thus,

$$\frac{1}{X}\psi(p) = -i \int\limits_{-\infty}^{p} \frac{\psi(q)dq}{1+\beta q^2} - \frac{c[\psi]}{\alpha}. \tag{9}$$

Substituting eigenfunction into (8) we obtain

$$c[\psi_\epsilon] = -\lim_{p\to-\infty}(p^2+\epsilon)\psi_\epsilon(p) = -C_\epsilon e^{\frac{-i\alpha\pi}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)}}. \tag{10}$$

We require that $\frac{1}{X}$ to be an Hermitian operator on the set of eigenfunctions:

$$\left\langle \frac{1}{X}\psi_{\epsilon_1} \middle| \psi_{\epsilon_2} \right\rangle = \left\langle \psi_{\epsilon_1} \middle| \frac{1}{X}\,\psi_{\epsilon_2} \right\rangle. \tag{11}$$

As a result

$$\sin[g(\epsilon_1) - g(\epsilon_2)] = 0, \tag{12}$$

where

$$g(\epsilon) = \frac{\alpha\pi}{2(\sqrt{\epsilon} + \sqrt{\beta}\epsilon)}.$$

Fixing one eigenvalue we obtain condition on the rest

$$\frac{\alpha}{2(\sqrt{\epsilon} + \sqrt{\beta}\epsilon)} = \delta_0 + n, \tag{13}$$

Let us analyze case of $\delta_0 = 0$ more thoroughly. Quantization condition reads

$$\frac{\alpha}{2(\sqrt{\epsilon}+\sqrt{\beta}\epsilon)} = n, \tag{14}$$

$$E_n = -\epsilon = -\frac{\left(1-\sqrt{1+\frac{2\alpha}{n}\sqrt{\beta}}\right)^2}{4\beta} \simeq -\frac{\alpha^2}{4n^2}+\frac{\alpha^3}{4n^3}\sqrt{\beta}-\frac{5\alpha^4}{16n^4}\beta, \tag{15}$$

where $n = 1, 2, \ldots$.

In general $\delta_0 \neq 0$ and

$$E_n = -\frac{\left(1-\sqrt{1+\frac{2\alpha}{n+\delta_0}\sqrt{\beta}}\right)^2}{4\beta}, \quad n = 0, 1, 2, \ldots \tag{16}$$

Results

- The first correction to the spectrum is proportional to $\sqrt{\beta}$

- There exist several spectral families. It means that deformation does not help to avoid singularity.