Bohr-Sommerfeld quantization rule in noncommutative space

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Several independent lines of theoretical physics investigations (**string theory**[#1], **black holes**[#2]) imply

$$\Delta X \geq \frac{\hbar}{2}\left(\frac{1}{\Delta P} + \beta \Delta P\right). \tag{1}$$

It means that $\Delta X \geq \Delta X_{min} = \hbar\sqrt{\beta}$. Kempf[#3] proposed to modify canonical commutation relation $[x, p] = i\hbar$

$$[X, P] = i\hbar(1 + \beta P^2). \tag{2}$$

Heisenberg inequality for this commutation relation is equivalent to (1).

[#1] Gross D. J., Mende P. F. String theory beyond the Planck scale // Nucl. Phys. B.— 1988.— V. 303, P. 407–454

[#2] Maggiore M. A generalized uncertainty principle in quantum gravity // Phys. Lett. B.— 1993.— V. 304, P. 65–69

[#3] Kempf A. et al. Hilbert space representation of the minimal length uncertainty relation // Phys. Rev. D.— 1995.— V. 52, P. 1108–1118

Known approximate methods

1. Perturbation theory[#4].
2. ?

Bohr-Sommerfeld quantization rule

To find spectrum of the problem

$$\left[\frac{P^2}{2m} + U(X)\right]\psi = E\psi, \quad [X, P] = i\hbar f(P). \tag{3}$$

Quasi-coordinate representation

$$X = x,\ P = P(p),\ p = -i\hbar\frac{d}{dx}; \quad \frac{dP(p)}{dp} = f(P).$$

Eigenfunction and action of P^2 on it

$$\psi(x) = \exp\left[\frac{i}{\hbar}S(x)\right], \quad P^2\psi = \left[P^2(S') - \frac{i\hbar}{2}[P^2(S')]''S'' + \ldots\right]\psi.$$

[#4] Brau F. Minimal length uncertainty relation and the hydrogen atom // J. Phys. A.— 1999.— V. 32, P. 7691–7696

In linear approximation over $\hbar$

$$\psi(x) = \frac{1}{\sqrt{|Pf(P)|}} \left(C_1 \exp\left[\frac{i}{\hbar}\int^x pdx\right] + C_2 \exp\left[-\frac{i}{\hbar}\int^x pdx\right]\right),$$

$P(p) = \sqrt{2m(E-U(x))}$. Matching

$$\int_{x_1}^{x_2} pdx = \pi\hbar(n+\delta), \quad n = 0, 1, 2, \ldots$$

or in initial variables

$$-\oint \frac{XdP}{f(P)} = 2\pi\hbar(n+\delta). \tag{4}$$

WKB approximation is valid if $\qquad P^2 \gg \hbar \left| \frac{d}{dx} Pf(P) \right|.$

For $f(P) = 1 + \beta P^2$

$$a \gg \lambda \gg \frac{\Delta X_{min}^2}{a}.$$

a — characteristic size of the system, $\lambda = 2\pi\hbar/P$.

1D **Examples** $[X,P]=i\hbar(1+\beta P^2).$

Hamiltonian	Eigenvalues E_n	$E_n - E_n^{\mathrm{WKB}}$
P^2+X^2	$(2n+1)+\beta\left(n^2+n+\frac{1}{2}\right)+O(\beta^2)$ #5	$\frac{1}{4}\beta+O(\beta^2)$
$P^2,\ -a<X<a$	$\left(\frac{\pi n}{2a}\right)^2+\frac{2}{3}\beta\left(\frac{\pi n}{2a}\right)^4+O(\beta^2)$ #6	$O(\beta^2)$
$P^2-\frac{\alpha}{X}$	$-\frac{1}{4\beta}\left(1-\sqrt{1+\frac{2\alpha}{n+\delta}\sqrt{\beta}}\right)^2$ #7	0
$P^2-\frac{\gamma}{X^2}$	—	$-\frac{4}{\beta}e^{-\pi(n+\delta)/\sqrt{\gamma}}$

#5 Kempf A. et al. Hilbert space representation of the minimal length uncertainty relation // Phys. Rev. D.— 1995.— V. 52, P. 1108–1118

#6 Detournay S. et al. About maximally localized states in quantum mechanics // Phys. Rev. D.— 2002.— V. 66.— 125004

#7 Fityo T. V. et al. One dimensional Coulomb-like problem in deformed space with minimal length // J. Phys. A.— 2006.— V. 39.— P. 2143–2149.

3D **Examples** $[X_i, P_j] = i\hbar((1+\beta P^2)\delta_{ij} + \beta' P_i P_j)$. $L^2 \to (l+1/2)^2$.

$$H = P^2 - \frac{\gamma}{X}.$$

Its spectrum in linear approximation over β, β' is [#8]

$$E_{n,l} \approx \underbrace{-\frac{\gamma^2}{4n^2} + \frac{\gamma^4}{8n^3}\left(\beta\left[\frac{2}{l+1/2} - \frac{1}{n}\right] + \beta'\left[\frac{1}{l+1/2} - \frac{1}{n}\right]}_{\text{WKB approximation}} + \frac{\beta - \beta'/2}{l(l+1)(l+1/2)}\right).$$

$$H = P^2 + X^2$$

$$E_{n,l} \approx \underbrace{2n + 3 + (\beta+\beta')(n+3/2)^2 + (\beta-\beta')(l+1/2)^2}_{\text{WKB approximation}} + 2\beta - \frac{\beta'}{2}.$$ [#9]

[#8] Benczik S. et al. The hydrogen atom with minimal length // Phys. Rev. A.— 2005.— V. 72.— 012104

[#9] Chang L. N. et al. Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations // Phys. Rev. D.— 2002.— V. 65.— 125027

Let us consider more general case of deformed space

$$[X, P] = i\hbar f(X, P).$$

It is unknown if we can express initial operators

$$X = X(\hat{x}, \hat{p}), \qquad P = P(\hat{x}, \hat{p}),$$

where $[\hat{x}, \hat{p}] = i\hbar$. Classical variables x and p always exist that

$$\{X, P\}_{x,p} = \frac{\partial X}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial x}\frac{\partial X}{\partial p} = f(X, P).$$

$$\oint p dx = \int\limits_{H \le E_n} dp dx =$$

$$= \int\limits_{H(P,X) \le E_n} \frac{dX dP}{f(X, P)} = 2\pi\hbar(n + \delta). \quad (5)$$

Example

Let us consider harmonic oscillator eigenvalue problem

$$(P^2 + X^2)\psi = E\psi$$

in deformed space

$$[X, P] = i(1 + \alpha X^2 + \beta P^2).$$

WKB approximation (5) gives

$$E_n = \underline{\underline{\frac{(\sqrt{\alpha}+\sqrt{\beta})^2}{4\alpha\beta} e^{2(n+\delta)\sqrt{\alpha\beta}}}} + \frac{(\sqrt{\alpha}-\sqrt{\beta})^2}{4\alpha\beta} e^{-2(n+\delta)\sqrt{\alpha\beta}} - \frac{\alpha+\beta}{2\alpha\beta}.$$

Leading term is underlined. It coincides with leading term of exact solution [#10].

[#10] Quesne C., Tkachuk V. M. Harmonic oscillator with nonzero minimal uncertainties in both position and momentum in a SUSYQM framework // J. Phys. A.— 2003.— V. 36.— 10373–10389.